# PHYS-448 Introduction to Particle Accelerators
# Solution of Tutorial 2

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**Exercise 1.** ALBA is a Synchrotron Light facility near Barcelona, Spain. The stored electron beam ($m_e = 511\,\mathrm{keV}/c^2$) has a Lorentz factor of $\gamma = 5870.841$. Knowing that the bending radius of the dipole magnets is $\rho = 7.05\,\mathrm{m}$, calculate their magnetic field strength $B$.

From the $\gamma$ factor we can obtain the energy of the electrons as follows:

$$E = \gamma m_e c^2 = 5870.841 \cdot \frac{0.511\,\mathrm{MeV}}{c^2} \cdot c^2 = 3000\,\mathrm{MeV} = 3\,\mathrm{GeV}$$

At such energy the momentum of the electrons $pc \approx E$ indeed:

$$pc = \sqrt{E^2 - m_e^2 c^4} = \sqrt{(3 \times 10^3\,\mathrm{MeV})^2 - (0.511\,\mathrm{MeV})^2} \approx 3\,\mathrm{GeV}.$$

We can obtain the dipole strength $B$ by applying the formula:

$$B\rho\,[\mathrm{T} \cdot \mathrm{m}] = \frac{1}{0.299\,79}\,p\,[\mathrm{GeV}/c] \Rightarrow B = 1.42\,\mathrm{T} \quad .$$

**Exercise 2.** The Proton Synchrotron Booster (PSB) at CERN accelerates protons up to a kinetic energy of $E_{\mathrm{kin}} = 1.4\,\mathrm{GeV}$. Are the protons *ultra-relativistic* at this stage? Calculate the momentum $p$ of the protons and the bending radius $\rho$ of the dipole magnets given that their maximum magnetic field strength is $B = 1.064\,\mathrm{T}$.

As seen in the lecture,

$$E_{\mathrm{kin}} = (\gamma - 1) m_p c^2$$

Hence, the relativistic $\gamma$ factor is:

$$\gamma = \frac{E_{\mathrm{kin}}}{m_p c^2} + 1 = \frac{1.4\,\mathrm{GeV}}{0.938\,\mathrm{GeV}} + 1 = 2.493$$

and the relativistic $\beta$ factor is:

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.916$$

the protons at this stage of acceleration are not yet ultra-relativistic, as $\gamma \gg 1$ does not yet apply. However, they are relativistic, since $\beta > \sqrt{3}/2 \approx 0.866$, i.e. $\gamma > 2$.

The proton momentum is:

$$p = \beta\gamma m_p c = 2.493 \cdot 0.916 \cdot 0.938\,\mathrm{GeV}/c^2 \cdot c = 2.142\,\mathrm{GeV}/c$$

and the bending radius of the dipole magnets is:

$$\rho = \frac{1}{0.29979}\frac{p[\mathrm{GeV}/c]}{B} = \frac{1}{0.29979}\frac{2.142\,\mathrm{GeV}/c}{1.064\,\mathrm{T}} = 6.71\,\mathrm{m}.$$

**Exercise 3.** A characteristic example of length contraction and time dilation is represented by the muon, which has an average life time of $\tau \simeq 2.2\,\mu\mathrm{s}$. Cosmic muons are produced in the upper layers of the Earth's atmosphere by the interaction of primary cosmic rays with atmospheric nuclei. Despite their short life time, the muons travelling at a speed close to the speed of light can reach the Earth's surface. The height of the atmosphere is $L \approx 15\,\mathrm{km}$. Assuming that the velocity of a muon is $v = 0.9992c$, calculate the muon life time measured by an observer on the Earth. What is the distance seen by the muon to reach the ground?

The relativistic gamma factor of the muon is:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.9992^2}} = 25$$

For an observer on the Earth, the time results to be dilated and the life time measured by them is:

$$\Delta t = \gamma \Delta t' = 25 \cdot 2.2\,\mu\mathrm{s} = 55\,\mu\mathrm{s}$$

On the other hand, the muon sees the earth moving towards itself. The distance $\Delta s$ it needs to travel appears to be shorter:

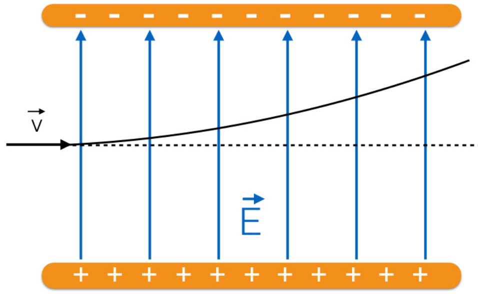
$$\Delta s = \frac{\Delta s'}{\gamma} = \frac{15\,\mathrm{km}}{25} = 600\,\mathrm{m}$$

This is a concrete manifestation of Einstein's theory of Special Relativity. Thanks to that, indeed the muons can reach the Earth.

**Exercise 4.** Consider a uniform electric field generated by two electrodes in the vacuum. A charged particle located at the edge of one electrode has initial velocity equal to zero. Undergoing the acceleration of the electric field, it arrives at the opposite electrode with a relativistic $\gamma = 5$. Calculate the required electrostatic potential difference (neglecting the gravitational force) for electrons ($m_e$=0.511 MeV/$c^2$ ), for protons ($m_p$=0.938 GeV/$c^2$), and lead ions ($m_{Pb}$=193.7 GeV/$c^2$, singly ionized).

The difference between initial and final energy has to be equal to the work of the electric force given by the potential difference between the electrodes multiplied by the particle charge:

$$E_f - E_i = q\Delta V$$
$$m\gamma_f c^2 - m\gamma_i c^2 = q\Delta V$$




where $\gamma_i = 1$ because the initial velocity of the particle is equal to zero. Therefore,

$$mc^2(\gamma - 1) = q\Delta V$$

the kinetic energy of the particle has to be equal to the work of the electric force.
Making the calculations for the three different particles:

$$\begin{aligned} q\Delta V^e &= m_e c^2 \cdot 4 = 0.511\,\mathrm{MeV} \cdot 4 = 2.04\,\mathrm{MeV} \\ q\Delta V^p &= m_p c^2 \cdot 4 = 0.938\,\mathrm{GeV} \cdot 4 = 3.75\,\mathrm{GeV}. \\ q\Delta V^{Pb} &= m_{Pb} c^2 \cdot 4 = 193.7\,\mathrm{GeV} \cdot 4 = 774.8\,\mathrm{GeV} \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta V^e &= m_e c^2 \cdot 4/q = 0.511\,\mathrm{MeV} \cdot 4/q = 2.04\,\mathrm{MV} \\ \Delta V^p &= m_p c^2 \cdot 4/q = 0.938\,\mathrm{GeV} \cdot 4/q = 3.75\,\mathrm{GV}. \\ \Delta V^{Pb} &= m_{Pb} c^2 \cdot 4/q = 193.7\,\mathrm{GeV} \cdot 4/q = 774.8\,\mathrm{GV} \end{aligned}$$

Due to the different masses, at the same relativistic $\gamma$ factor the required electrostatic potential difference increases as the particle mass. The Van de Graaff generators in tandem like this one at Brookhaven National Labs have a setup that produces 30 MV. Even for these devices it is impossible to reach such kinetic energy for protons or ions for which you have to build an accelerator as the one we study in this course.

**Exercise 5.** Imagine you bend a proton by using an electric field as illustrated in the figure above. What is the electric field strength you would need to achieve a bending radius equivalent to that of a 8.3 T dipole magnet for a proton energy of $E = 3.3\,\mathrm{TeV}$. We approximate that the electric field is always perpendicular to the direction of motion.

The bending radius produced by the dipole is given by the beam regidity formula:

$$\rho = \frac{p[\mathrm{GeV}/c]}{0.29979 \cdot B} = \frac{3300\,\mathrm{GeV}/c}{0.29979 \cdot 8.3\,\mathrm{T}} = 1326.2\,\mathrm{m}.$$

By equating the Lorentz force to the centripetal force:

$$\vec{F_L} = \vec{F_c}$$
$$q(\vec{E} + \vec{v} \times \vec{B}) = q \cdot E = \frac{p[\mathrm{eV}]\beta c}{\rho}$$

where has been considered only an electric field acting on the particle and perpendicular to its velocity.

$$E = \frac{p[\mathrm{eV}]\beta c}{q \cdot \rho} \approx \frac{3300\,\mathrm{GeV}/c \cdot c}{e \cdot 1326.2\,\mathrm{m}} = 2.49\,\mathrm{GV/m}.$$

That is a huge electric field value.

**Exercise 6.** Will an electron have the same kinetic energy as a proton when accelerated through the same voltage drop of $\Delta V = 1\,\mathrm{MV}$? (Don't forget to reverse the polarity of the voltage source.) What about the momentum and velocity?
(Old CRT-type) color TV sets have an accelerating potential of 26 kV. Is the electron beam that hits the screen relativistic?

Yes they will have the same kinetic energy but different momentum and velocity:

$$E_k = (\gamma - 1)mc^2 = q\Delta V = 1\,\mathrm{MeV}$$
$$\gamma_p = \frac{q\Delta V + m_p c^2}{m_p c^2} = \frac{1 \times 10^{-3}\,\mathrm{GeV} + 0.938\,\mathrm{GeV}/c^2 \cdot c^2}{0.938\,\mathrm{GeV}/c^2 \cdot c^2} = 1.00107$$
$$\gamma_e = \frac{q\Delta V + m_e c^2}{m_e c^2} = \frac{1\,\mathrm{MeV} + 0.511\,\mathrm{MeV}/c^2 \cdot c^2}{0.511\,\mathrm{MeV}/c^2 \cdot c^2} = 2.95695$$

Therefore,

$$\beta_p = \sqrt{1 - 1/\gamma_p^2} = 0.04614$$
$$\beta_e = \sqrt{1 - 1/\gamma_e^2} = 0.94108$$

and the momentum:

$$p_p = \beta_p \gamma_p m_p c = 43.3256\,\mathrm{MeV}/c$$
$$p_e = \beta_e \gamma_e m_e c = 1.4219\,\mathrm{MeV}/c$$
$$\frac{p_p}{p_e} = 30.47.$$

The electron will not be relativistic for an accelerating potential of 26 kV since $\gamma_e = 1.05$ .